Key Moments in the History of Numerical Analysis

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PART I

Broad historical outline

The prehistory of Numerical Analysis

In contrast to more classical fields of mathematics, like Analysis, Number Theory or Algebraic Geometry, Numerical Analysis (NA) became an independent mathematical discipline only in the course of the 20th Century.

This is perhaps surprising, given that effective methods of computing approximate numerical solutions to mathematical problems are already found in antiquity (well before Euclid!), and were especially prevalent in ancient India and China.

While algorithmic mathematics thrived in ancient Asia, classical Greek mathematicians showed relatively little interest in it and cultivated Geometry instead. Nevertheless, Archimedes (3rd Century BCE) was a master calculator.

The prehistory of Numerical Analysis (cont.)

Many numerical methods studied in introductory NA courses bear the name of great mathematicians including Newton, Euler, Lagrange, Gauss, Jacobi, Fourier, Chebyshev, and so forth.

However, it should be kept in mind that well into the 19th Century, the distinction between mathematics and natural philosophy (including physics, chemistry, astronomy etc.) was almost non-existent. Scientific specialization is a modern phenomenon, and nearly all major mathematicians were also physicists and astronomers.

Indeed, a look at the original works of these great mathematicians shows that almost without exception, the numerical methods bearing their names were introduced in the context of physical, astronomical or technical applications.

An example: Newton interpolation

In Book III, Lemma V, of his 1687 masterpiece, *Mathematical Principles of Natural Philosophy*, Isaac Newton (1642-1727) poses the following problem:

To find a curved line of the parabolic kind which shall pass through any given number of points.

(Note: here parabolic = polynomial).

After solving the problem by constructing the interpolating polynomial by divided differences, he proceeds to apply it in the subsequent Lemma VI to the problem:

Certain observed places of a comet being given, to find the place of the same at any intermediate given time.

An example: Newton interpolation (cont.)

Cor. 2. But their orbits will be so near to parabolas, that parabolas may be used for them without sensible error.

Cor. 3. And, therefore, by Cor. 7, Prop. XVI. Book 1, the velocity of every comet will always be to the velocity of any planet, supposed to be revolved at the same distance in a circle about the sun, nearly in the subduplicate proportion of double the distance of the planet from the centre of the sun to the distance of the comet from the sun's centre, very nearly. Let us suppose the radius of the *orbis magnus*, or the greatest semi-diameter of the ellipsis which the earth describes, to consist of 1000000 parts; and then the earth by its mean diurnal motion will describe 1720212 of those parts, and 71675½ by its horary motion. And therefore the comet, at the same mean distance of the earth from the sun, with a velocity which is to the velocity of the earth as $\sqrt{2}$ to 1, would by its diurnal motion describe 2432747 parts and 101354½ parts by its horary motion. But at greater or less distances both the diurnal and horary motion will be to this diurnal and horary motion in the reciprocal subduplicate proportion of the distances and is therefore given.

Cor. 4. Wherefore if the *latus rectum* of the parabola is quadruple of the radius of the *orbis magnus*, and the square of that radius is supposed to consist of 1000000000 parts, the area which the comet will daily describe by a radius drawn to the sun will be 1216373½ parts, and the horary area will be 50682½ parts. But, if the *latus rectum* is greater or less in any proportion, the diurnal and horary area will be less or greater in the subduplicate of the same proportion reciprocally.

LEMMA V.

To find a curve line of the parabolic kind which shall pass through any given number of points.

Let those points be A, B, C, D, E, F &c, and from the same to any right line HN, given in position, let fall as many perpendiculars AH, BI CK, DL, EM, FN, &c.

a $2a$ $3a$ $4a$ $5a$

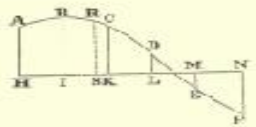
b $2b$ $3b$ $4b$

c $2c$ $3c$ $4c$

d $2d$ $3d$

e $2e$

f



Case 1. If HI, IK, KL, &c., the intervals of the points H, I, K, L, M N, &c., are equal, take a, $2b$, $3b$, $4b$, $5b$, &c., the first differences of the perpendiculars AH, BI, CK, &c.; their second differences c, $2c$, $3c$, $4c$, &c.; their third, d, $2d$, $3d$, &c., that is to say, so as AH — BI may be = a, BI

An example: Newton interpolation (cont.)

CK $-$ 2*b*, CK $-$ DL $-$ 3*b*, DL $+$ EM $-$ 4*b*, $-$ EM $+$ FN $-$ 5*b*, &c., then *b* $-$ 2*c* $=$ *c*, &c., and so on to the last difference, which is here *f*. Then, erecting any perpendicular ES which may be considered as an ordinate of the curve required, in order to find the length of this ordinate, suppose the intervals HI, IK, KL, LM &c. to be units, and let AH $=$ *a*, $-$ HS $=$ *p*, $\frac{1}{2}p$ into $-$ IS $=$ *q*, $\frac{1}{2}q$ into $+$ SK $=$ *r*, $\frac{1}{2}r$ into $+$ SL $=$ *s*, $\frac{1}{2}s$ into $-$ SM $=$ *t*; proceeding, to wit, to ME, the last perpendicular but one, and prefixing negative signs before the terms HS, IS, &c., which lie from S towards A; and affirmative signs before the terms SK, SL, &c., which lie on the other side of the point S; and observing well the signs, RS will be $=$ *a* $+$ *bp* $+$ *cq* $+$ *dr* $+$ *es* $+$ *ft*, $+$ &c.

Case 2. But if HI, IK, &c., the intervals of the points H, I, K, L, &c., are unequal, take *b*, 2*b*, 3*b*, 4*b*, 5*b*, &c. the first differences of the perpendiculars AH, BI, CK, &c., divided by the intervals between those perpendiculars; *c*, 2*c*, 3*c*, 4*c*, &c., their second differences, divided by the intervals between every two; *d*, 2*d*, 3*d*, &c., their third differences divided by the intervals between every three; *e*, 2*e*, &c., their fourth differences, divided by the intervals between every four; and so forth; that is, in such manner, that *b* may be $= \frac{AH - BI}{HI}$, $2b = \frac{BI - CK}{IK}$, $3b = \frac{CK - DL}{KL}$, &c., then $c = \frac{b - 2b}{HK}$, $2c = \frac{2b - 3b}{HL}$, $3c = \frac{3b - 4b}{KM}$, &c., then $d = \frac{c - 2c}{HL}$, $2d = \frac{2c - 3c}{LM}$, &c. And those differences being found, let AH be $=$ *a*, $-$ HS $=$ *p*, *p* into $-$ IS $=$ *q*, *q* into $-$ SK $=$ *r*, *r* into $+$ SL $=$ *s*, *s* into $+$ SM $=$ *t*; proceeding, to wit, to ME, the last perpendicular but one: and the ordinate RS will be $=$ *a* $+$ *bp* $+$ *cq* $+$ *dr* $+$ *es* $+$ *ft*, $+$ &c.

Cor. Hence the areas of all curves may be nearly found; for if some number of points of the curve to be squared are found, and a parabola be supposed to be drawn through those points, the area of this parabola will be nearly the same with the area of the curvilinear figure proposed to be squared: but the parabola can be always squared geometrically by methods vulgarly known.

LEMMA VI.

Certain observed places of a comet being given, to find the place of the same to any intermediate given time.

Let HI, IK, KL, LM (in the preceding Fig.), represent the times between the observations; HA, IE, KC, LD, ME, five observed longitudes of the comet; and HS the given time between the first observation and the longitude required. Then if a regular curve ABCDE is supposed to be drawn through the points A, B, C, D, E and the ordinate RS is found out by the preceding lemma, RS will be the longitude required.

The prehistory of Numerical Analysis (cont.)

Beginning with the second half of the 19th Century, mathematician becomes almost synonymous with pure mathematician. Most mathematics becomes conceptual and far removed from applications. Numerical methods are developed out of necessity to solve problems arising in geodesy, astronomy, physics and engineering. Examples:

- In Linear Algebra we have Gauss–Jordan elimination for $A\mathbf{x} = \mathbf{b}$ (Geodesy) and Jacobi's methods for $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \lambda\mathbf{x}$ (Celestial Mechanics);
- In Ordinary Differential Equations, we have the methods of Adams and Moulton (Celestial Mechanics) and Runge–Kutta (Aerodynamics);
- In PDEs, the Rayleigh–Ritz method (vibration problems in Acoustics and Elasticity).

The prehistory of Numerical Analysis (cont.)

This situation persists in the first few decades of the 20th Century. NA is not recognized as a mathematical discipline; university professorships and courses are non-existent, textbooks are scarce, and computing machines are primitive.

Early books on numerical methods are:

1. E. T. Whittaker and G. Robinson, *The Calculus of Observations. A Treatise on Numerical Mathematics*, London, 1924.
2. G. Cassinis, *Calcoli Numerici, Grafici e Meccanici*, Pisa, 1928.
3. J. B. Scarborough, *Numerical Mathematical Analysis*, Baltimore, 1930.

These books are mostly aimed at scientists and engineers.

The prehistory of Numerical Analysis (cont.)

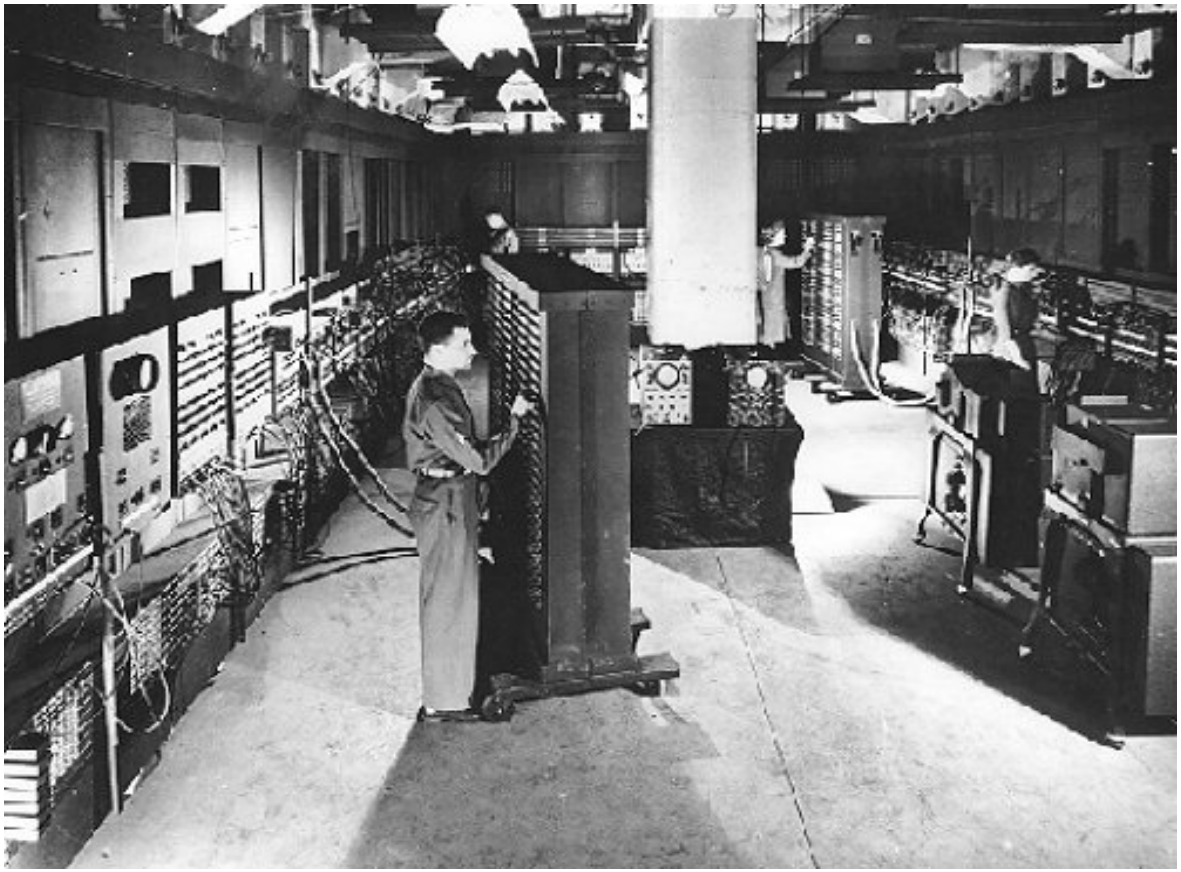
It should be kept in mind that before the invention of electronic digital computers (mid-1940s), computations were done by hand, by slide ruler, by desk calculators and, in the first part of the 20th Century, by electro-mechanical and analogical devices.

Two excellent references on the early history of NA and Computing are:

H. H. Goldstine, *A History of Numerical Analysis from the 15th Through the 19th Century*, Springer, New York, 1977.

H. H. Goldstine, *The Computer from Pascal to von Neumann*, Princeton University Press, Princeton, NJ, 1972.

The first large-scale electronic computer: ENIAC (1945)



ENIAC = Electronic Numerical Integrator and Computer

Key Moments of 20th Century Numerical Analysis

Modern NA begins in the 1940s, as a result of two related events: the participation of large numbers of mathematicians in the war effort (especially in USA, Germany, USSR), and the development of the first digital electronic computers.

However, some important pre-war developments should be mentioned here:

- The analysis of finite difference methods for PDEs: Richardson (1910); Phillips and Wiener (1923); Courant, Friedrichs and Lewy (1928).
- The emergence of a school of applied mathematics in Germany, mostly through the efforts of Felix Klein (Göttingen: Runge, Courant; Berlin: von Mises).
- The creation, in Italy, of the first research institute entirely devoted to NA (Picone, late 1920s).

Key Moments of 20th Century NA (cont.)

In the States, very little work in Applied Mathematics was done prior to the late 1930s. The situation changed dramatically as a result of the influx of European refugees, especially from Nazi Germany, beginning in 1933. Many of these scientists were to give a major contribution to the war effort.

The pre-eminent figure here is perhaps Richard Courant (1888–1972), who was the successor of David Hilbert (1862–1943) as the director of the famous Mathematical Institute in Göttingen.

In the period 1900–1930, the University of Göttingen had established itself as one of the great centers for research in mathematics and physics worldwide. During the 19th Century, the mathematics faculty included Gauss, Dirichlet, Riemann and Klein. In the 1920s, Göttingen became a leading center for research in quantum physics.

Richard Courant (1888–1972)

Key Moments of 20th Century NA (cont.)

Two famous papers on finite differences:

L. F. Richardson, *The approximate arithmetical solution by finite differences of physical problems involving differential equations with an application to stresses in a masonry dam*, Philosophical Transactions of the Royal Society of London, A, 210 (1910), pp. 307–357.

R. Courant, K. O. Friedrichs, and H. Lewy, *Über die partiellen Differenzengleichungen der mathematischen Physik*, Mathematische Annalen 100 (1928), pp. 32–74. Translated by Phyllis Fox as: *On the partial difference equations of mathematical physics*, IBM Journal of Research and Development, 11 (1967), pp. 215–234. With a commentary by P. D. Lax.

Key Moments of 20th Century NA (cont.)

The celebrated CFL paper used finite differences not to solve practical problems, but to establish existence and uniqueness results for elliptic boundary value and eigenvalue problems, and for the initial value problem for hyperbolic and parabolic PDEs. It also introduced for the first time the method of random walks for approximating the solution of PDEs.

This paper was to have enormous influence on the numerical analysis of PDEs, especially through the work of John von Neumann at Los Alamos at the time of the Manhattan Project.

The paper also influenced the development by John Curtiss in the early 1950s of Monte Carlo methods for evaluating (functionals of) solutions of PDEs.

A quote from the CFL paper

From the Introduction:

Nowhere do we assume the existence of the solution to the differential equation problem—on the contrary, we obtain a simple existence proof by using the limiting process. For the case of elliptic equations convergence is obtained independently of the choice of the mesh, but we will find that for the case of the initial value problem for hyperbolic equations, convergence is obtained only if the ratio of the mesh widths in different directions satisfies certain inequalities which in turn depend on the position of the characteristics relative to the mesh.

A quote from the CFL paper (cont.)

From Part II, section 2:

In the same way that the solution of a linear hyperbolic equation at a point S depends only on a certain part of the initial line—namely the "domain of dependence" lying between the two characteristics drawn through S, the solution of the difference equation has also at the point S a certain domain of dependence... We shall now consider a (...) general rectangular mesh with mesh size equal to h (time interval) in the t-direction and equal to kh (space interval) in the x-direction, where k is a constant. The domain of dependence for the difference equation (...) for this mesh will now either lie entirely within the domain of dependence of the differential equation, $\partial^2 u/\partial t^2 - \partial^2 u/\partial x^2 = 0$, or on the other hand will contain this latter region inside its own domain according as $k < 1$ or $k > 1$ respectively.

A quote from the CFL paper (cont.)

From this follows a remarkable fact: if for the case $k < 1$ *one lets* $h \to 0$*, then the solution to the difference equation in general* cannot *converge to the solution of the differential equation... That convergence does occur for the case* $k > 1$ *will be proved in Section 3.*

The authors explain that this is due to the fact that if $k < 1$, a change in the initial values of the solution of the differential equation at points of the $t = 0$ line between the two domains of dependence must change the solution of the differential equation, but cannot influence the solution of the difference equation.

This is, of course, the famous CFL condition.

Key Moments of 20th Century NA (cont.)

Another very influential paper by Courant appeared during the war:

R. Courant, *Variational methods for the solution of problems of equilibrium and vibrations*, Bull. Amer. Math. Soc., 69 (1943), pp. 1–23.

This paper contains many of the ideas of what later became known as finite element methods. The full development of FEMs, however, began around 1960 at the hand of aerospace and civil engineers (Felippa & Clough, UC Berkeley; Zinkiewicz and Irons, UK; Argyris, Germany).

Mathematically, the FEM is a modern development of the classical methods of Rayleigh, Ritz, and Galerkin.

A quote from Courant's 1943 paper

From Part III, p. 15:

If the variational problem contains derivatives not higher than the first order the method of finite differences can be subordinated to the Rayleigh-Ritz method by considering in the competition only functions ϕ which are linear in the meshes of a sub-division of our net into triangles formed by diagonals of the squares of the net. For such polyhedral functions the integrals become sums expressed by the finite number of values of ϕ in the net-points and the minimum condition becomes our difference equations. Such an interpretation suggests a wide generalization *which provides great flexibility and seems to have considerable practical value.*

A quote from Courant's 1943 paper (cont.)

Instead of starting with a quadratic or rectangular net we may consider from the outset any polyhedral surfaces with edges over an arbitrarily chosen *(preferably triangular)* net*. Our integrals again become finite sums, and the minimum condition will be equations for the values of* ϕ *in the net-points. While these equations seem less simple than the original difference equations, we gain the enormous advantage of better adaptability to the data of the problem and thus we can often obtain good results with very little numerical calculation.*

These are prophetic words indeed!

Key Moments of 20th Century NA (cont.)

Another key figure is that of John von Neumann, who came to the US in the early 1930s to join the newly formed Institute for Advanced Studies in Princeton, NJ. Other members ot the IAS included Albert Einstein and Hermann Weyl.

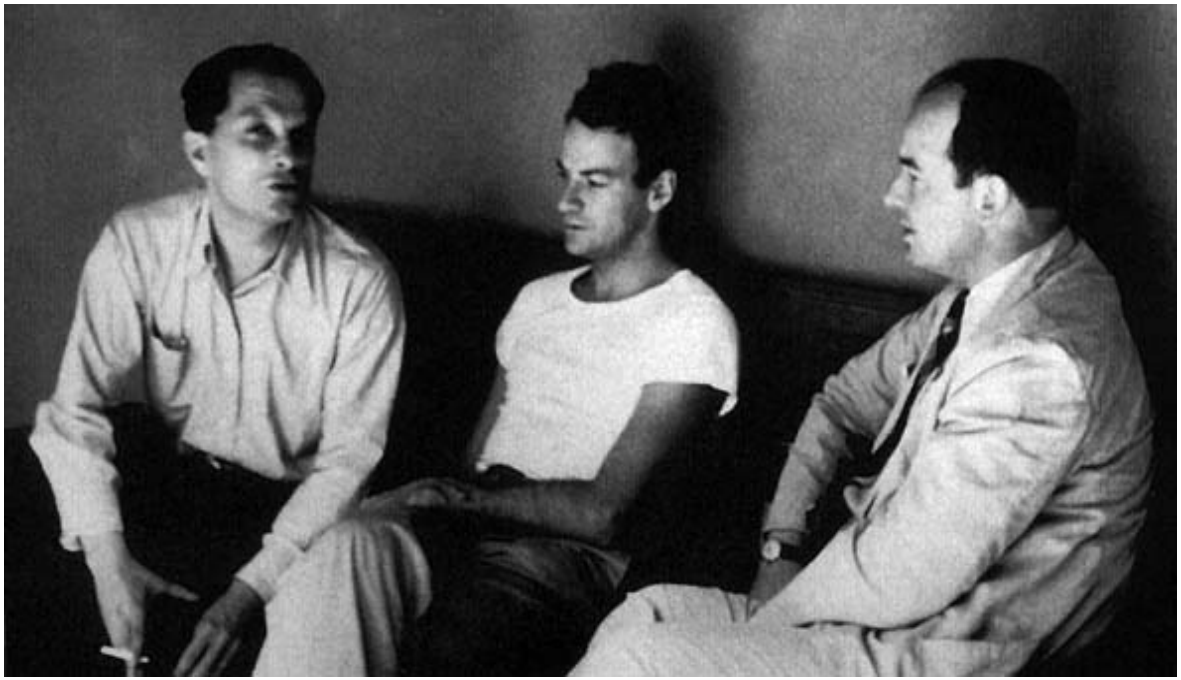
A Hungarian-born child prodigy, von Neumann initially worked with Hilbert on mathematical logic, the foundations of mathematics, and the rigorous formulation of quantum mechanics. He also worked on topological groups, rings of operators, ergodic theory and many other areas of pure mathematics.

In applied mathematics he created the field of game theory (with the economist Oskar Morgenstern) and did fundamental work in hydrodynamics, meteorology, numerical linear algebra, and numerical PDEs. He is, however, best known for his work on the logical design of digital computers and his theory of automata.

John von Neumann (1903–1957)

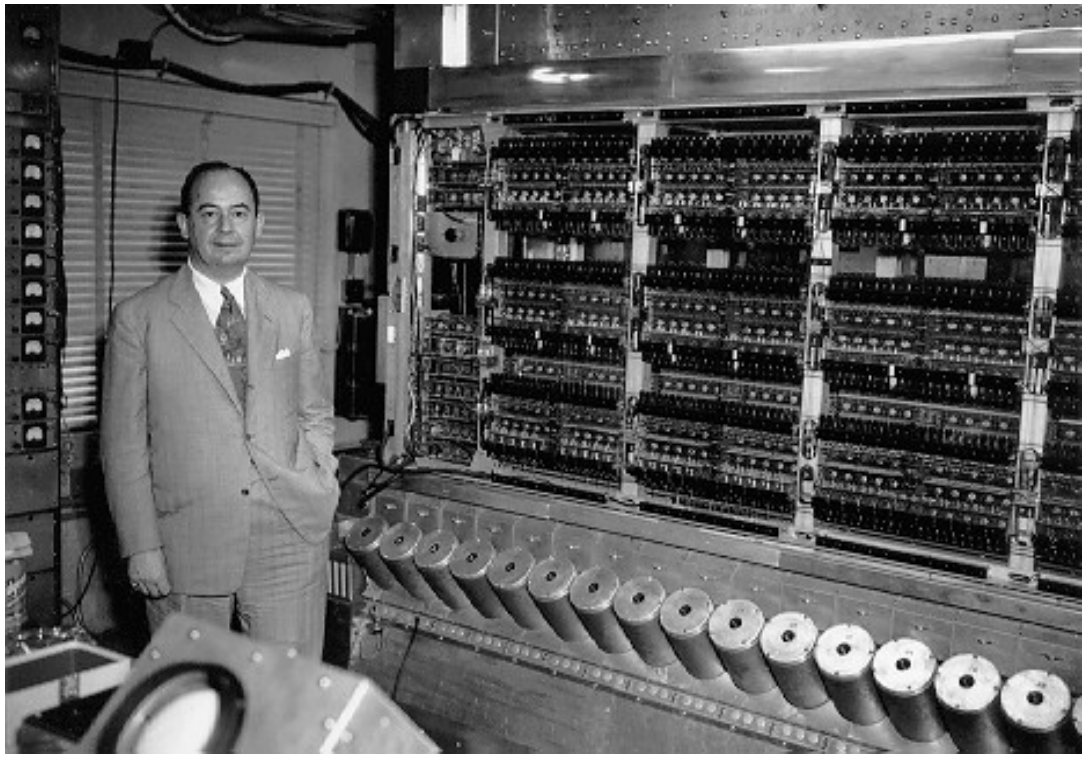
A giant of 20th Century mathematics.

Stan Ulam, Richard Feynman and John von Neumann (Bandelier, near Los Alamos, NM, late 1940s)



Three key figures in 20th Century science.

von Neumann' IAS Computer (Princeton, 1952)



A clone of this machine was installed at Los Alamos

(MANIAC = Mathematical Analyzer, Numerical Integrator and Computer)

Key Moments of 20th Century NA (cont.)

Among von Neumann's contributions to NA and scientific computing we mention:

1. The well-known von Neumann stability condition (based on Fourier analysis) for the time-integration of evolution problems
2. Pioneering work (mostly with H. Goldstine) in numerical linear algebra
3. Early contributions to linear programming (later developed by Dantzig and others)
4. The development (with Metropolis and Ulam) of the Monte Carlo method

Owing to his enormous prestige as a great mathematician, von Neumann is often cited as one of the main factors making NA acceptable as a *bona fide* mathematical discipline (and not just a collection of recipes).

Key Moments of 20th Century NA (cont.)

J. von Neumann and H. H. Goldstine, *Numerical inverting of matrices of high order*, Bull. Amer. Math. Soc., 53 (1947), pp. 1021–1099.

This long paper is considered by many to mark the beginning of modern numerical linear algebra. It is mostly devoted to a detailed round-off error analysis of matrix factorization and inversion methods. Among other things, it introduces the notion of condition number $\kappa(A)$ of an SPD matrix with respect to inversion. The paper also marks the first use of backward error analysis, later greatly developed and widely applied by Wilkinson.

At about the same time (1948), a paper by Alan Turing provided a similar analysis of Gaussian elimination.

A quote from von Neumann and Goldstine (1947)

From the Preface:

The purpose of this paper is to derive rigorous error estimates in connection with the inverting of matrices of high order. The reasons for taking up this subject at this time in such considerable detail are essentially these: First, the rather widespread revival of mathematical interest in numerical methods, and the introduction of new procedures and devices which make it both possible and necessary to perform operations of this type on matrices of much higher orders than was even remotely practical in the past. Second, the fact that considerably diverging opinions are now current as to the extremely high or extremely low precisions which are required when inverting matrices of order $n \geq 10$.

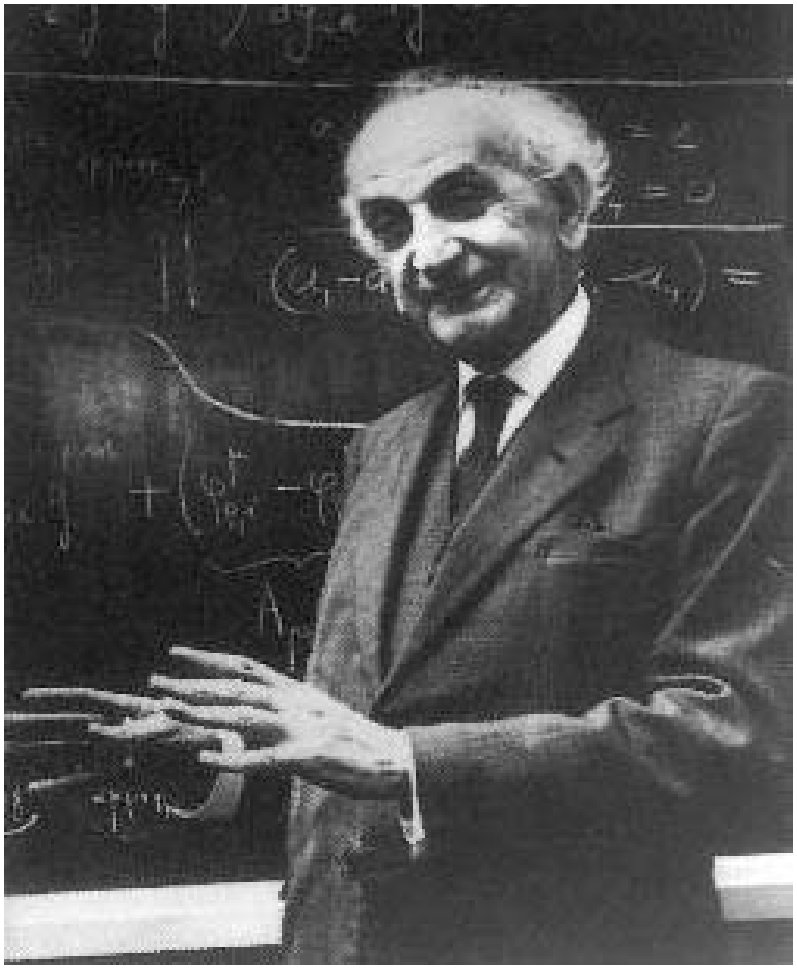
Key Moments of 20th Century NA (cont.)

Numerical linear algebra also received great impetus from the establishment on the UCLA campus of the Institute for Numerical Analysis of the National Bureau of Standards (1947-1954).

Among the main figures associated with INA we mention George Forsythe, Cornelius Lanczos, Isaac Schoenberg, Olga Taussky-Todd, John Todd, Magnus Hestenes, and Eduard Stiefel.

Both the Lanczos algorithm and the Conjugate Gradient method date to this exciting period. Their widespread use, however, will have to wait until the 1970s.

Cornelius Lanczos (1893–1974)



Magnus Hestenes and Eduard Stiefel

Co-discoverers of the Conjugate Gradient Method.

Key Moments of 20th Century NA (cont.)

Other notable figures in numerical linear algebra in the 1950s and 1960s include Wallace Givens (Argonne), Alston Householder (Oak Ridge), Jim Wilkinson (NPL, UK), Heinz Rutishauser (ETH Zurich), Alexander Ostrowski (Basel), John Francis (UK), and Vera Kublanovskaya (USSR); stable and efficient methods for linear systems and eigenvalue calculations are due to them.

Important contributions to the area of iterative methods in this period are due to David Young, Richard Varga, and Gene Golub.

In the mid-60s, fundamental work by Golub, Velvel Kahan and others led to stable algorithms for SVD-based least-squares and pseudoinverse calculations.

1964 Gatlinburg Conference

Emilie Haynsworth, Alan Hoffman, Olga Taussky-Todd and Gene Golub.

Key Moments of 20th Century NA (cont.)

The 1950s also see great progress in the analysis of finite difference methods for PDEs (including the famous equivalence theorem of Peter Lax) and the development of Alternating Direction Implicit (ADI) and splitting-up methods by Peaceman, Rachford, Douglas, Birkhoff, Varga, Young in USA and by Yanenko and others in USSR.

Together with the SOR method, ADI schemes are the method of choice for the numerical solution of discretized PDEs during much of the 1950s and 1960s. Only in the 1970s will the Conjugate Gradient method gain favor, especially after the development of incomplete Cholesky preconditioning. Much credit here goes to John Reid (UK), Koos Meijerink and Henk van der Vorst (Netherlands), and to a 1976 paper by Gene Golub, Paul Concus and Dianne O'Leary.

Key Moments of 20th Century NA (cont.)

In the 1960s and 1970s a class of (non-iterative) methods for solving the Poisson equation on regular grids in near-optimal time is developed by Buneman, Golub, Buzbee, Nielson and others. These techniques are closely related to cyclic reduction and the Fast Fourier Transform (FFT) of Cooley and Tuckey (1965).

These fast Poisson solvers will later be superseded by variants of the multigrid method, first studied in USSR by Fedorenko and Bakhvalov in the 1960s, and later by Achi Brandt, Wolfgang Hackbusch, and many others.

Chief applications of PDE solvers during this period are in the areas of nuclear reactor modeling and petroleum engineering.

Key Moments of 20th Century NA (cont.)

The 1980s and 1990s see rapid developments in the field of Krylov subspace methods for nonsymmetric linear systems (Young, Saylor, Sonneveld, Eisenstat, Elman, Schulz, Saad, Freund, Nachtigal, van der Vorst), preconditioning, multilevel algorithms, and large-scale eigenvalue solvers.

On the theory side, the important Faber-Manteuffel Theorem (1984) settles a question of Golub on the existence of optimal Krylov methods based on short recurrences.

We also mention the impact of parallel computing, stimulating the development of domain decomposition schemes by Lions, Widlund, and many others.

Additional references and resources

G. Birkhoff, *Solving Elliptic Problems: 1930–1980*, in M. H. Schulz, Ed., *Elliptic Problem Solvers*, Academic Press, NY, 1981.

C. Brezinski and L. Wuytack, Eds., *Numerical Analysis: Historical Developments in the Twentieth Century*, North-Holland, Amsterdam, 2001.

M. R. Hestenes and J. Todd, *Mathematicians Learning to Use Computers. The Institute for Numerical Analysis, UCLA, 1947–1954*. National Institute of Standards and Technology and Mathematical Association of America, Washington, DC, 1991.

S. G. Nash, Ed., *A History of Scientific Computing*, ACM Press and Addison-Wesley Publishing Co., NY, 1990.

The SIAM History Project at http://history.siam.org contains a number of articles and transcripts of interviews—highly recommended!

PART II

The early history of matrix iterations

Iterative methods prior to about 1930

The earliest reference to an iterative approach to solving $A\mathbf{x} = \mathbf{b}$ appears to be contained in a letter by Gauss to his student Gerling dated 26 December 1823, in the context of solving least squares problems via the normal equations.

After briefly describing his method (essentially a relaxation procedure) on a 4×4 example, Gauss wrote:

You will in future hardly eliminate directly, at least not when you have more than two unknowns. The indirect procedure can be done while one is half asleep, or is thinking about other things.

Cf. *Werke, IX*, p. 278. See also E. Bodewig, *Matrix Calculus*, 1956.

Iterative methods prior to about 1930 (cont.)

In 1826 Gauss gave a block variant of the method in the *Supplementum* to his famous work on least squares, *Theoria Combinationis Observationum Erroribus Minimis Obnoxiae* (English translation by Pete Stewart published by SIAM in 1995).

Solution of normal equations by iteration became standard in 19th Century Germany, especially among geodesists and astronomers (including Gerling, Bessel, Schumacher,...).

According to Bodewig, Gauss had to solve systems with 20-30-40 unknowns. These systems were diagonally dominant and convergence was fast. In 1890, Nagel used iteration to solve a system of 159 unknowns arising in the triangulation of Saxony.

Iterative methods prior to about 1930 (cont.)

In 1845 Jacobi introduced his own iterative method, again for solving normal equations for least squares problems arising in astronomical calculations.[1]

In the same paper he makes use of carefully chosen plane rotations to increase the diagonal dominance of the coefficient matrix. This is perhaps the first occurrence of preprocessing of a linear system in order to speed up the convergence of an iterative method. He gives a 3×3 example.

Only in a subsequent paper (1846) he will use plane rotations to diagonalize a symmetric matrix.

[1] *Über eine neue Auflösungsart der bei der Methode der kleinsten Quadrate vorkommenden linearen Gleichungen*, Astronomische Nachrichten, 22 (1845), 297–306. Reprinted in *Gesammelte Werke*, vol. III, pp. 469–478.

Jacobi's example

Jacobi takes for his example a linear system that appears in Gauss' *Theoria Motus Corporum Coelestium in Sectionibus Conicis Solem Ambientium* (1809).

In modern notation, Jacobi wants to solve $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 27 & 6 & 0 \\ 6 & 15 & 1 \\ 0 & 1 & 54 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 88 \\ 70 \\ 107 \end{bmatrix}.$$

Jacobi uses a plane rotation (with angle $\alpha = 22^{\circ}\,30'$) to annihilate the (1,2)-(2,1) coefficient. After this, the transformed system is solved in three iterations of Jacobi's method. Each iteration adds about one digit of accuracy.

Iterative methods prior to about 1930 (cont.)

Again in the context of least squares, in 1874 another German, Seidel, publishes his own iterative method. The paper contains what we now (inappropriately) call the Gauss–Seidel method, which he describes as an improvement over Jacobi's method.

Seidel notes that the unknowns do not have to be processed cyclically (in fact, he advises *against* it!); instead, one could choose to update at each step the unknown with the largest residual. He seems to be unaware that this is precisely Gauss' method.

In the same paper, Seidel mentions a block variant of his scheme. He also notes that the calculations can be computed to variable accuracy, using fewer decimals in the first iterations. His linear systems had up to 72 unknowns.

Iterative methods prior to about 1930 (cont.)

Another important 19th Century development that is worth mentioning were the independent proofs of convergence by Nekrasov (1885) and by Pizzetti (1887) of Seidel's method for systems of normal equations (more generally, SPD systems).

These authors were the first to note that a necessary and sufficient condition for the convergence of the method (for an arbitrary initial guess $\mathbf{x}^0$) is that all the eigenvalues of the iteration matrix must satisfy $|\lambda| < 1$. Nekrasov and Mehmke (1892) also gave examples to show that convergence can be slow.

Nekrasov seems to have been the first to relate the rate of convergence to the dominant eigenvalue of the iteration matrix. The treatment is still in terms of determinants, and no use is made of matrix notation.

Iterative methods prior to about 1930 (cont.)

In the early 20th century we note the following important contributions:

- The method of Richardson (1910);
- The method of Liebmann (1918).

These papers mark the first use of iterative methods in the solution of finite difference approximations to elliptic PDEs.

Richardson's method is still well known today, and can be regarded as an acceleration of Jacobi's method by means of over- or under-relaxation factors. Liebmann's method is identical with Seidel's.

Iterative methods prior to about 1930 (cont.)

The first systematic treatment of iterative methods for solving linear systems appeared in a paper by the famed applied mathematician Richard von Mises and his collaborator (and later wife) Hilda Pollaczek-Geiringer, titled *Praktische Verfharen der Gleichungsauflösung* (ZAMM **9**, 1929, pp. 58–77).

This was a rather influential paper. In it, the authors gave conditions for the convergence of Jacobi's and Seidel's methods, the notion of diagonal dominance playing a central role.

The paper also studied a stationary version of Richardson's method, which will be later called Mises' method by later authors.

Richard von Mises and Hilda Pollaczek-Geiringer

Two pioneers in the field of iterative methods.

Mauro Picone and the INAC

The paper by von Mises and Pollaczek-Geiringer was read and appreciated by Mauro Picone (1885-1977) and his collaborators at the *Istituto Nazionale per le Applicazioni del Calcolo* (INAC).

The INAC was among the first institutions in the world entirely devoted to the development of numerical analysis. Both basic research and applications were pursued by the INAC staff.

The Institute had been founded in 1927 by Picone, who was at that time professor of Infinitesimal Analysis at the University of Naples. In 1932 Picone moved to the University of Rome, and the INAC was transferred there. It became a CNR institute in 1933.

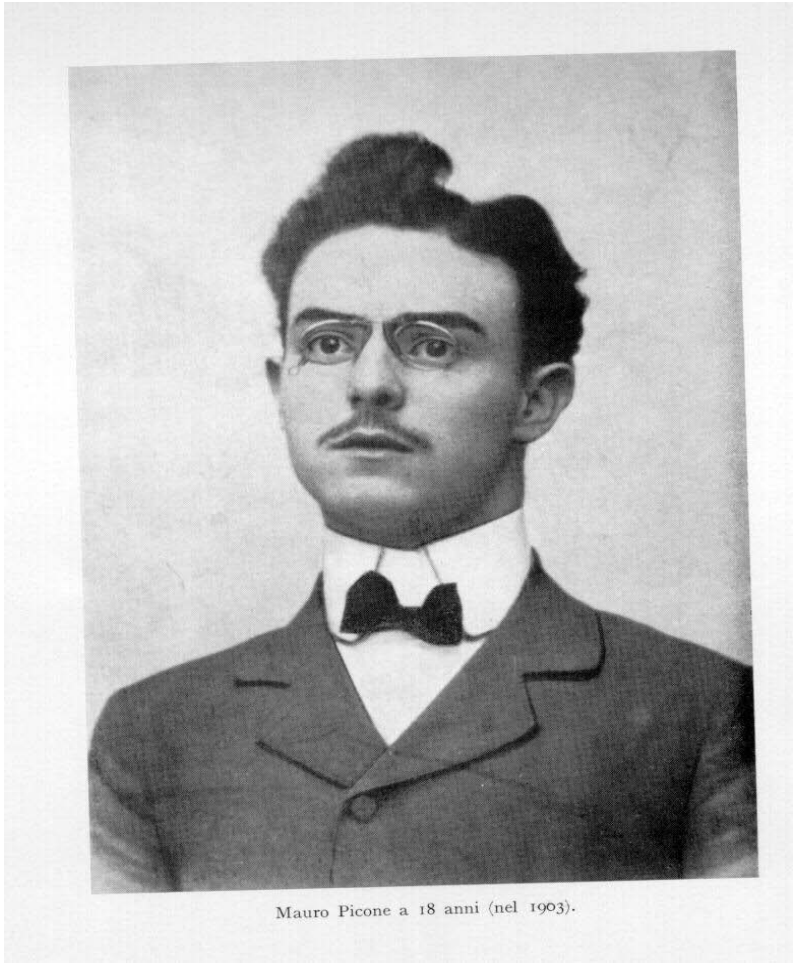
Mauro Picone and the INAC (cont.)

Picone's interest in numerical computing dated back to his service as an artillery officer in WWI, when he was put to work on ballistic tables.

The creation of the INAC required great perseverance and political skill on his part, since most of the mathematical establishment of the time was either indifferent or openly against it.

Picone's own training was in classical ("hard") analysis. He worked mostly on the theory of differential equations and in the calculus of variations. He was also keenly interested in constructive and computational functional-analytic methods.

Mauro Picone in 1903



Mauro Picone a 18 anni (nel 1903).

Mauro Picone around 1950

Mauro Picone and the INAC (cont.)

Throughout the 1930's and beyond, under Picone's direction, the INAC employed a number of young mathematicians, many of whom later became very well known.

INAC researchers did basic research and also worked on a large number of applied problems supplied by industry, government agencies, and the Italian military. Picone was fond of saying that

$$Matematica\ Applicata = Matematica\ Fascista$$

In the 1930s the INAC also employed up to eleven *computers* and *draftsmen*. These were highly skilled men and women who were responsible for carrying out all the necessary calculations using a variety of mechanical, electro-mechanical, and graphical devices.

Mauro Picone and the INAC (cont.)

As early as 1932, Picone designed and taught one of the first courses on numerical methods ever offered at an Italian university (the course was called *Calcoli Numerici e Grafici*). The course was taught in the School of Statistical and Actuarial Sciences, because Picone's colleagues in the Mathematics Institute denied his request to have the course listed among the electives for the degree in Mathematics.

The course covered root finding, maxima and minima, solutions of linear and nonlinear systems, interpolation, numerical quadrature, and practical Fourier analysis.

Both Jacobi's and Seidel's method are discussed (including block variants). Picone's course was not very different from current introductory classes in numerical analysis.

Mauro Picone and the INAC (cont.)

In the chapter on linear systems, Picone wrote:

The problem of solving linear systems has enormous importance in applied mathematics, since nearly all computational procedures lead to such problem. The problem can be considered as a multivariate generalization of ordinary division... While for ordinary division there exist automatic machines, things are not so for the solution of linear systems, although some great minds have set themselves this problem in the past. Indeed, the first researches go back to Lord Kelvin. Recently, Professor Mallock of Cambridge University has built a highly original electrical machine which can solve systems of $n \leq 10$ *equations in as many unknowns. This and other recent efforts, however, are far from giving a practical solution to the problem.*

Lamberto Cesari's work on iterative methods

Lamberto Cesari (Bologna, 1910; Ann Arbor, MI, 1990) studied at the *Scuola Normale* in Pisa under L. Tonelli, then in 1934 went to Germany to specialize under the famous mathematician C. Carathéodory. After his return from Germany, Cesari joined the INAC in Rome.

While at INAC Cesari wrote the paper *Sulla risoluzione dei sistemi di equazioni lineari per approssimazioni successive* (La Ricerca Scientifica, **8**, 1937, pp. 512–522.)

Cesari later became a leading expert in various branches of mathematical analysis and optimization. In 1948 he emigrated to the US, where he had a brilliant career, first at Purdue, then at Michigan.

Lamberto Cesari (1910-1990)

Lamberto Cesari's 1937 paper

Cesari gives a general theory of stationary iterations in terms of matrix splittings. Having written the linear system as

$$\omega A\mathbf{x} = \omega\mathbf{b}, \quad \omega \neq 0\,,$$

he introduces the splitting

$$\omega A = B + C, \quad \det(B) \neq 0\,,$$

and proves that the stationary iteration associated with this splitting converges for any choice of the initial guess if and only if the smallest (in magnitude) root of

$$\det(B + \lambda C) = 0$$

is greater than 1. This is equivalent to the usual condition $\rho(B^{-1}C) < 1$.

Lamberto Cesari's 1937 paper (cont.)

In the same paper, Cesari applies his general theory to the methods of Jacobi, Seidel, and von Mises (stationary Richardson). He uses $\omega \neq 1$ only for the latter.

In the case of von Mises' method (analyzed for the SPD case), Cesari notes that, regardless of ω, the rate of convergence of the method deteriorates as the ratio of the extreme eigenvalues of A increases. He writes:

In practice, we found that already for $\frac{\lambda_{\max}(A)}{\lambda_{\min}(A)} > 10$ *the method of von Mises converges too slowly.*

This observation leads Cesari to the idea of polynomial preconditioning.

Lamberto Cesari's 1937 paper (cont.)

Given estimates $a \approx \lambda_{\min}(A)$ and $b \approx \lambda_{\max}(A)$, Cesari determines the coefficients of the polynomial $p(x)$ of degree k such that the ratio of the maximum and the minimum of $q(x) = xp(x)$ is minimized over $[a, b]$, for $1 \leq k \leq 4$.

The transformed system

$$p(A)A\mathbf{x} = p(A)\mathbf{b}\,,$$

which he shows to be equivalent to the original one, can be expected to have a smaller condition number.

Cesari ends the paper with a brief discussion of when this approach may be useful and gives the results of numerical experiments with all three methods on a 3×3 example using a polynomial of degree $k = 1$.

Lamberto Cesari's 1937 paper (cont.)

Cesari's paper was not without influence: it is cited, sometimes at length, in important papers by Forsythe (1952-1953) and in the books by Bodewig (1956), Faddeev & Faddeeva (1960), Householder (1964), Wachspress (1966) and Saad (2003) among others.

It is, however, not cited in the influential books of Varga (1962) and Young (1971).

Cesari's paper is important for our story also because of the effect it had on a former student and assistant of Picone.

Cimmino's method

Gianfranco Cimmino (Naples, 1908; Bologna, 1989) graduated at Naples with Picone in 1927 with a thesis on approximate solution methods for the heat equation in 2D.

After a period spent at INAC and a study stay in Germany (again with Carathéodory), he undertook a brilliant academic career. He became a full professor in 1938, and in 1939 moved to the chair of Mathematical Analysis at Bologna, where he spent his entire career.

Cimmino's work was mostly in analysis: theory of linear elliptic PDEs, calculus of variations, integral equations, functional analysis, etc. He also wrote 5-6 short papers on matrix computations.

Gianfranco Cimmino (1908-1989)

Cimmino's method (cont.)

In 1938 *La Ricerca Scientifica* published a short (8 pages) paper by Cimmino, titled *Calcolo approssimato per le soluzioni dei sistemi lineari*.

As Picone himself explained in a brief introductory note, after reading Cesari's paper Cimmino reminded him of a method that he had developed around 1932 while he was at INAC. Cimmino's method did not appear to fit under Cesari's "systematic treatment," yet

it is most worthy of consideration in the applications because of its generality, its efficiency and, finally, because of its guaranteed convergence which can make the method practicable in many cases. Therefore, I consider it useful to publish in this journal Prof. Cimmino's note on the above mentioned method, note that he has accepted to write upon my insistent invitation.

Cimmino's method (cont.)

Cimmino considers the system $A\mathbf{x} = \mathbf{b}$ where A is a real $n \times n$ matrix, initially assumed to be nonsingular.

If $\mathbf{a}_i^T = [a_{i1}, a_{i2}, \ldots, a_{in}]$ denotes the ith row of A, the solution $\mathbf{x}_* = A^{-1}\mathbf{b}$ is the unique intersection point of the n hyperplanes described by

$$\langle \mathbf{a}_i, \mathbf{x} \rangle = b_i, \quad i = 1, 2, \ldots, n\,. \tag{1}$$

Given an initial approximation $\mathbf{x}^{(0)}$, Cimmino takes, for each $i = 1, 2, \ldots, n$, the reflection (mirror image) $\mathbf{x}_i^{(0)}$ of $\mathbf{x}^{(0)}$ with respect to the hyperplane (1):

$$\mathbf{x}_i^{(0)} = \mathbf{x}^{(0)} + 2\,\frac{b_i - \langle \mathbf{a}_i, \mathbf{x}^{(0)} \rangle}{\|\mathbf{a}_i\|^2}\,\mathbf{a}_i\,. \tag{2}$$

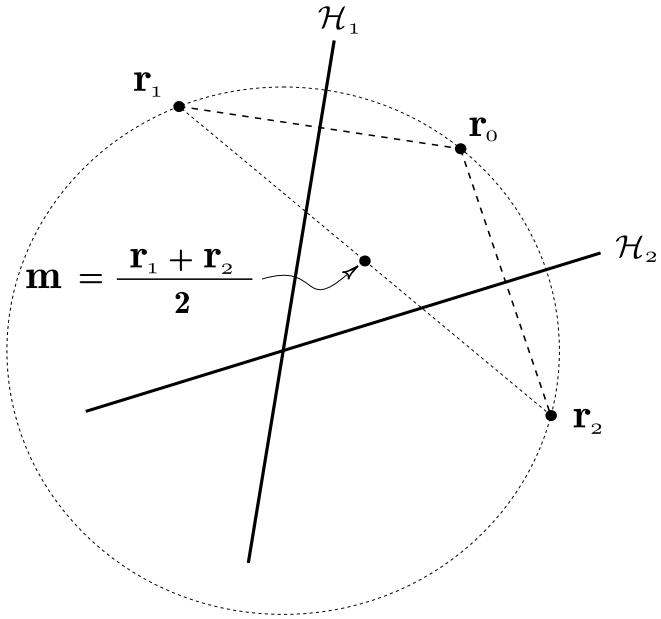
Cimmino's method (cont.)

Given n arbitrarily chosen positive quantities $m_1, \ldots, m_n$, Cimmino constructs the next iterate $\mathbf{x}^{(1)}$ as the center of gravity of the system formed by placing the n masses m_i at the points $\mathbf{x}_i^{(0)}$ given by (2), for $i = 1, 2, \ldots, n$. Cimmino notes that the initial point $\mathbf{x}^{(0)}$ and its reflections with respect to the n hyperplanes (1) all lie on a hypersphere the center of which is precisely the point common to the n hyperplanes, namely, the solution of the linear system. Because the center of gravity of the system of masses $\{m_i\}_{i=1}^n$ must necessarily fall inside this hypersphere, it follows that the new iterate $\mathbf{x}^{(1)}$ is a better approximation to the solution than $\mathbf{x}^{(0)}$:

$$\|\mathbf{x}^{(1)} - \mathbf{x}_*\| < \|\mathbf{x}^{(0)} - \mathbf{x}_*\| .$$

The procedure is then repeated starting with $\mathbf{x}^{(1)}$.

Cimmino's method ($n = 2$)



From C. D. Meyer, *Matrix Analysis and Applied Linear Algebra*, SIAM, 2000.

Cimmino's method (cont.)

Cimmino proves that his method is always convergent.

In the same paper Cimmino shows that the iterates converge to a solution of $A\mathbf{x} = \mathbf{b}$ even in the case of a singular (but consistent) system, provided that rank $(A) \geq 2$.

He then notes that the sequence $\{\mathbf{x}^{(k)}\}$ converges even when the linear system is inconsistent, always provided that rank $(A) \geq 2$. Much later (1967) Cimmino wrote:

The latter observation, however, is just a curiosity, being obviously devoid of any practical usefulness. [sic!]

It can be shown that for an appropriate choice of the masses m_i, the sequence $\{\mathbf{x}^{(k)}\}$ converges to the minimum 2-norm solution of $\|\mathbf{b} - A\mathbf{x}\|_2 = \min$.

Cimmino's method (cont.)

In matrix form, Cimmino's method can be written as follows:

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \frac{2}{\mu} A^T D(\mathbf{b} - A\mathbf{x}^{(k)})$$

($k = 0, 1, \ldots$), where

$$D = \text{diag}\left(\frac{m_1}{\|\mathbf{a}_1\|^2}, \frac{m_2}{\|\mathbf{a}_2\|^2}, \ldots, \frac{m_n}{\|\mathbf{a}_n\|^2}\right)$$

and $\mu = \sum_{i=1}^{n} m_i$.

Therefore, Cimmino's method is a special case of von Mises' method (stationary Richardson) on the normal equations if we let $m_i = \|\mathbf{a}_i\|^2$. Cimmino's method corresponds to using $\omega = 2/\mu$ for the relaxation factor. With such a choice, convergence is guaranteed.

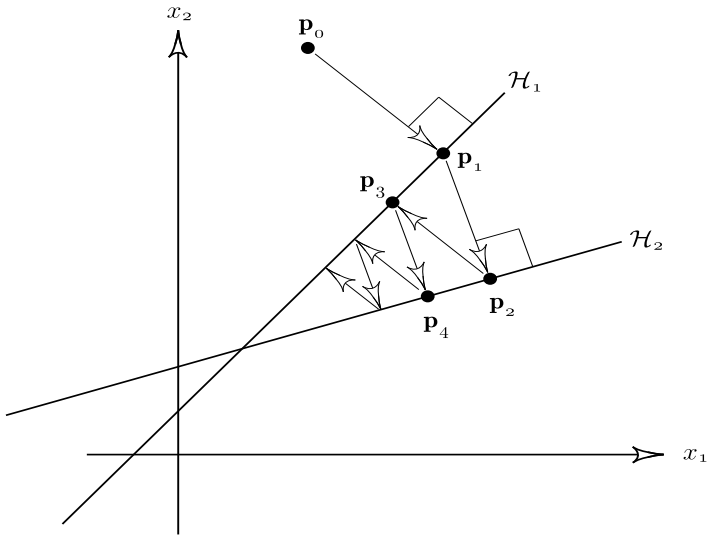
Cimmino's legacy

Cimmino's method, like the contemporary (and related) method of Kaczmarz, did not attract much attention until many years later.

Although it was described by Forsythe (1953) and in the books of Bodewig (1956), Householder (1964), Gastinel (1966) and others, I was able to find only 8 journal citations of Cimmino's 1938 paper until 1980.

After 1980, the number of papers and books citing Cimmino's (as well as Kaczmarz's) method picks up dramatically, and it is now in the hundreds. Moreover, both methods have been reinvented several times.

Kaczmarz's method ($n = 2$)



From C. D. Meyer, *Matrix Analysis and Applied Linear Algebra*, SIAM, 2000.

Cimmino's legacy (cont.)

Two major reasons for this surge in popularity are the fact that the method has the regularizing property when applied to discrete ill-posed problems, and the high degree of parallelism of the algorithm.

Today, Cimmino's method is rarely used to solve linear systems. Rather, it forms the basis for algorithms that are used to solve systems of inequalities (the so-called convex feasibility problem), and it has applications in computerized tomography, radiation treatment planning, medical imaging, etc.

Indeed, most citations occur in the medical physics literature, an outcome that would have pleased Gianfranco Cimmino.

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